LESSON

6 Derivatives

Straight lines may be ideal to human beings, but most functions have curved graphs. This does not stop us from projecting straight lines on them! For example, at the point marked x on the graph in Figure 6.1, the function is clearly increasing. However, exactly how fast is the function increasing at that point? Since "how fast" refers to a slope, we draw in the *tangent line*, the line straight through the point that heads in the same direction as the curve (see Figure 6.2). The slope of the tangent line tells us how fast the function is increasing at the given point.

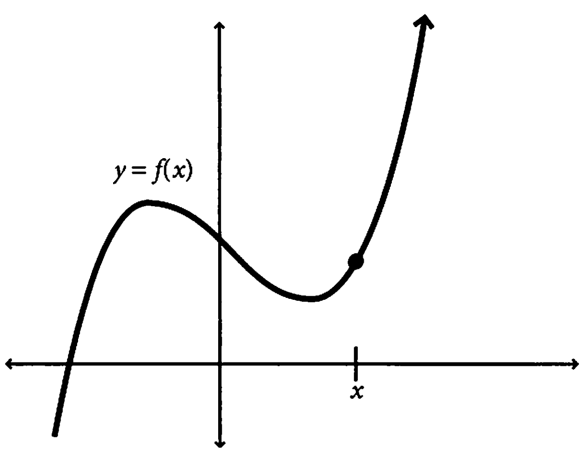

Figure 6.1

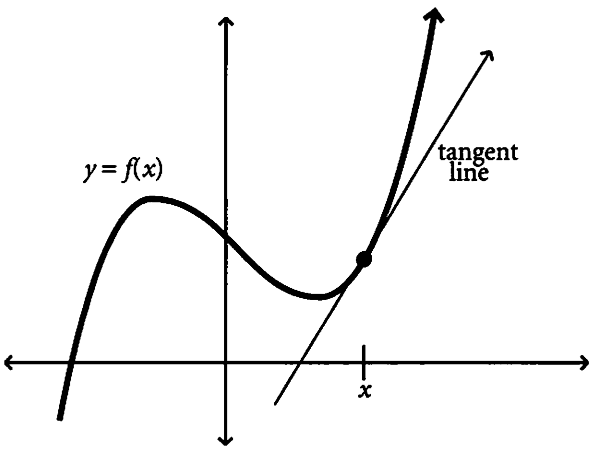

Figure 6.2

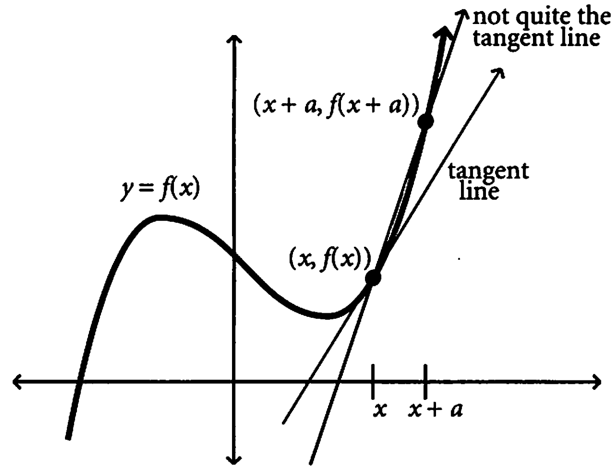

Figure 6.3

We can figure out the y-value of this point by plugging x into f and getting $(x, f(x))$. However, we can't get the slope of the tangent line when we have just one point. To get a second point, we go ahead a little further along the graph (see Figure 6.3). If we go ahead by distance a, the second point will have an x-value of $x + a$ and a y-value of $f(x + a)$.

Because this second point is on the curve and not on the tangent line, we get a line that is not quite the tangent line. Still, its slope will be close to the one we want, so we calculate as follows:

$$slope = \frac{f(x + a) - f(x)}{(x + a) - x} = \frac{f(x + a) - f(x)}{a}$$

To make things more accurate, we pick a second point that is closer to the first one by using a smaller a. This is depicted in Figure 6.4.

In fact, if we take the limit as a goes to zero, we will get the slope of the tangent line exactly. This is called the *derivative* of $f(x)$ and is written $f'(x)$.

$$f'(x) = \lim_{a \to 0} \frac{f(x + a) - f(x)}{a}$$

$$= \text{slope of the tangent line at point } (x, f(x))$$

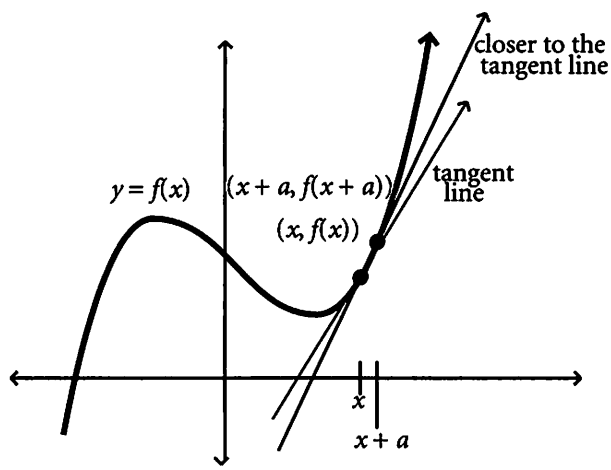

Figure 6.4

Example

What is the derivative of $f(x) = x^2$?

Solution

Start with the definition of the derivative.

$$f'(x) = \lim_{a \to 0} \frac{f(x + a) - f(x)}{a}$$

Use $f(x) = x^2$.

$$f'(x) = \lim_{a \to 0} \frac{(x + a)^2 - x^2}{a}$$

Multiply out and simplify.

$$f'(x) = \lim_{a\to 0} \frac{\cancel{x^2} + 2ax + a^2 - \cancel{x^2}}{a}$$

Factor and simplify.

$$f'(x) = \lim_{a\to 0} \frac{(2x + a)\cancel{a}}{\cancel{a}}$$

Plug in for the limit.

$$f'(x) = \lim_{a\to 0} 2x + a = 2x$$

The derivative is $f'(x) = 2x$. This means that the slope at any point on the curve $y = x^2$ is exactly twice the x-coordinate. The situation at $x = -2$, $x = 0$, and $x = 1$ is shown in Figure 6.5.

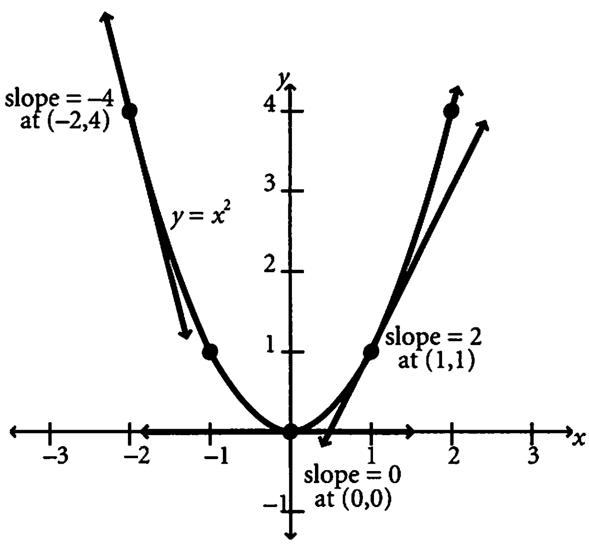

Figure 6.5

Example

What is the slope of the line tangent to $g(x) = \sqrt{x}$ at $x = 9$?

Solution

Start with the definition of the derivative.

$$g'(x) = \lim_{a\to 0} \frac{g(x + a) - g(x)}{a}$$

Use $g(x) = \sqrt{x}$.

$$g'(x) = \lim_{a\to 0} \frac{\sqrt{x + a} - \sqrt{x}}{a}$$

Rationalize the numerator.

$$g'(x) = \lim_{a\to 0} \left(\frac{\sqrt{x + a} - \sqrt{x}}{a}\right)\left(\frac{\sqrt{x + a} + \sqrt{x}}{\sqrt{x + a} + \sqrt{x}}\right)$$

Multiply and simplify.

$$g'(x) = \lim_{a\to 0} \frac{\cancel{x} + a + \cancel{\sqrt{x}\cdot\sqrt{x+a}} - \cancel{\sqrt{x}\cdot\sqrt{x+a}} - \cancel{x}}{a(\sqrt{x + a} + \sqrt{x})}$$

Simplify.

$$g'(x) = \lim_{a\to 0} \frac{\cancel{a}}{\cancel{a}(\sqrt{x + a} + \sqrt{x})}$$

Plug in to evaluate the limit.

$$g'(x) = \lim_{a\to 0} \frac{1}{\sqrt{x + a} + \sqrt{x}} = \frac{1}{\sqrt{x + 0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

The derivative of $g(x) = \sqrt{x}$ is thus $g'(x) = \frac{1}{2\sqrt{x}}$. This means that at $x = 9$, the slope of the tangent line is $g'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$. This is illustrated in Figure 6.6.

Example

Find the equation of the tangent line to $h(x) = 2x^2 - 5x + 1$ at $x = 3$.

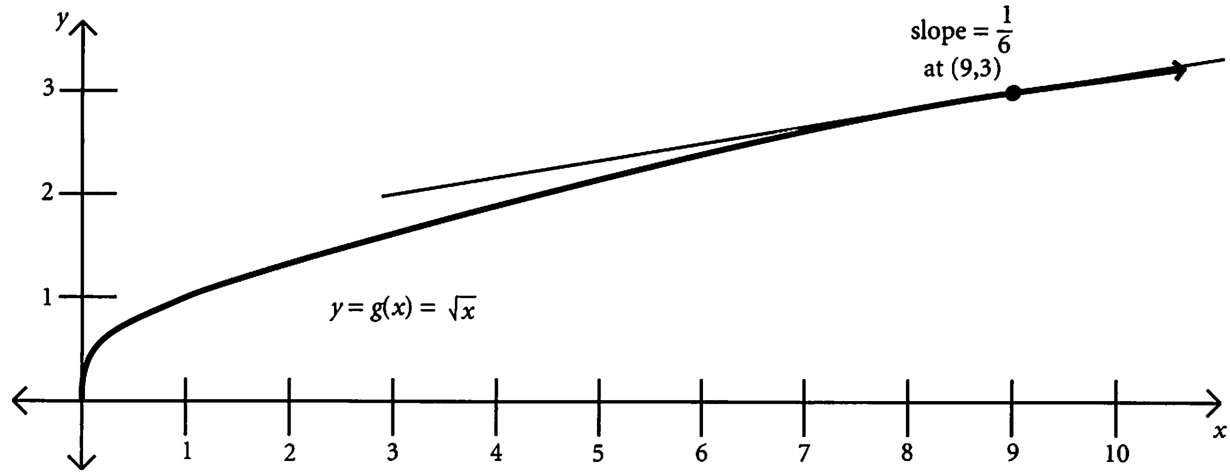

Figure 6.6

Solution

To find the equation of the tangent line, we need a point and a slope. The y-value at $x = 3$ is $h(3) = 2(3)^2 - 5(3) + 1 = 4$, so the point is (3,4). And to get the slope, we need the derivative. Start with the definition of the derivative.

$$h'(x) = \lim_{a \to 0} \frac{h(x + a) - h(x)}{a}$$

Use $h(x) = 2x^2 - 5x + 1$.

$$h'(x) = \lim_{a \to 0} \frac{2(x + a)^2 - 5(x + a) + 1 - (2x^2 - 5x + 1)}{a}$$

Multiply out and simplify.

$$h'(x) = \lim_{a \to 0} \frac{\cancel{2x^2} + 4ax + 2a^2 - \cancel{5x} - 5a + \cancel{1} - \cancel{2x^2} + \cancel{5x} - \cancel{1}}{a}$$

Factor out and simplify.

$$h'(x) = \lim_{a \to 0} \frac{(4x + 2a - 5)\cancel{a}}{\cancel{a}}$$

Evaluate the limit.

$$h'(x) = \lim_{a \to 0} 4x + 2a - 5 = 4x - 5$$

Thus, the derivative of $h(x) = 2x^2 - 5x + 1$ is $h'(x) = 4x - 5$. The slope at $x = 3$ is $h'(3) = 4(3) - 5 = 7$. The equation of the tangent line is therefore $y = 7(x - 3) + 4 = 7x - 17$. This is shown in Figure 6.7.

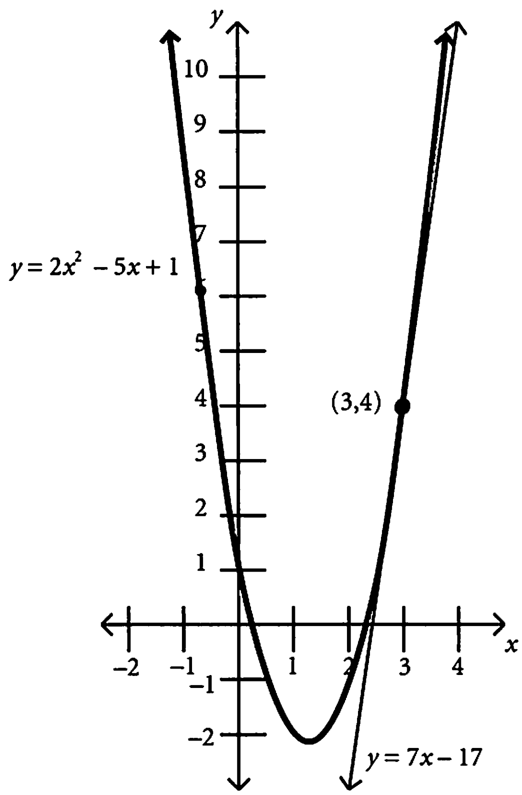

Figure 6.7

▶ Practice

1. Find the derivative of $f(x) = 8x + 2$.

2. If $h(x) = x^2 + 5$, then what is $h'(x)$?

3. Find the derivative of $g(x) = 10$.

4. What is the derivative of $g(x) = 3 - 5x$?

5. Find the derivative of $f(x) = 3\sqrt{x}$.

6. If $k(x) = x^3$, then what is $k'(x)$?

7. Find the slope of $f(x) = 3x^2 + x$ at $x = 2$.

8. Where does the graph of $g(x) = x^2 - 4x + 1$ have a slope of 0?

9. Find the equation of the tangent line to $h(x) = 1 - x^2$ at $(2,-3)$.

10. What is the equation of the tangent line of $k(x) = 5x^2 + 2x$ at $x = 1$?